JOURNAL OF DIFFERENTIAL EQUATIONS 94, 83–94 (1991)

$C^{1,1}$-Regularity of Constrained Area Minimizing Hypersurfaces

PETER STERNBERG* AND WILLIAM P. ZIEMER†

Mathematics Department, Indiana University, Bloomington, Indiana 47405

AND

GRAHAM WILLIAMS‡

Mathematics Department, Wollongong University, Australia

Received February 26, 1990

In [SWZ] we considered the constrained least gradient problem

$$\inf\left\{\int_\Omega |\nabla u|\,dx\colon u\in C^{0,1}(\bar\Omega),\ |\nabla u|\leqslant 1 \text{ a.e.},\ u=g \text{ on } \partial\Omega\right\}, \tag{1}$$

where $\Omega\subset R^n$ is an open set and g is a continuous function on $\partial\Omega$ satisfying the Lipschitz condition $|g(p)-g(q)|\leqslant|p-q|$ for all $p, q\in\partial\Omega$. This study was motivated by work of Kohn and Strang [KoS] who showed that (1) arises as the relaxation of a nonconvex variational problem in optimal design. The main purpose of [SWZ] was to show that the solution u to (1) can be explicitly constructed by equating the set $\{u\geqslant t\}$ with the solution to the obstacle problem

$$\inf\{P(E,\Omega)\colon \bar\Omega\supset E\supset L,\ \bar E\cap(M)^i=\varnothing\}, \tag{2}$$

where $P(E,\Omega)$ denotes the perimeter of E in Ω and where L and M are closed sets that depend on t and satisfy an interior ball condition of radius R. That is, for each $x\in L$, we assume there is a ball $B\subset L$ of radius $r\geqslant R$ such that $x\in B$, and similarly for M. We also assume that $L\cap(M)^i=\varnothing$.

This naturally led us to the question of regularity of solutions of (2). The

* Research supported in part by a grant from the National Science Foundation and an Indiana University Summer Faculty Fellowship.

† Research supported in part by a grant from the National Science Foundation.

‡ Research conducted while visiting Indiana University.

0022-0396/91 $3.00

strongest previous result on this question was due to Tamanini [T] who showed that a minimizer E of (2) satisfies

$$\partial E \in C^{1,1/2}$$

in a neighborhood of $\partial L \cup \partial M$. In this paper we show that this can be improved to $C^{1,1}$ regularity. We note that this is essentially optimal in light of the example in [SWZ, Fig. 3]. The example shows that ∂E is not of class C^2. Our techniques are fashioned after those that appear in [CR].

In the following, we discuss some preliminary regularity properties of the boundary of the obstacles L and M. Near any $p \in \partial E \cap \partial(L \cup M) \cap \Omega$, we know from [T] that ∂E can be represented as the graph of a $C^{1,1/2}$ function u. We take a coordinate system of the form (x, y), valid for $|x| < \delta$, with $p = (0, 0)$, where $x \in R^{n-1}$ lies in the tangent plane to the graph of u at 0 and where $y \in R^1$. Here δ depends on the $C^{1,1/2}$-norm of u. For convenience of notation, we also take $\nabla u(0) = 0$. By taking δ smaller if necessary, we may assume that L lies below the graph of u and that M lies above it.

We begin with an elementary proposition which states that $\partial L \cap \Omega$ and $\partial M \cap \Omega$ are locally graphs of Lipschitz functions.

LEMMA 1. *Let $x_0 = (0, 0) \in \partial E \cap \partial L \cap \Omega$. Then there exist constants $\delta = \delta(R)$ and $K = K(R)$ such that for $|x| < \delta$, ∂L can be represented as the graph of a Lipschitz function f with Lipschitz constant K. Similarly, near $x_0 \in \partial E \cap \partial M \cap \Omega$, ∂M can be represented as the graph F of a Lipschitz function.*

Proof. As stated above, we can introduce a coordinate system centered at $x_0 \in \partial E \cap \partial L$, where ∂E is representable as the graph of a $C^{1,1/2}$-function $u = u(x)$, $x \in R^{n-1}$. Furthermore, we may assume the coordinate chosen so that

$$u(0) = \nabla u(0) = 0.$$

Hence, there exist constants C_1 and $\delta > 0$ depending only on $\|u\|_{C^{1,1/2}}$ such that

$$|u(x)| \leqslant C_1 |x|^{3/2}$$

for $|x| < \delta$. Let

$$f(x) = \sup\{y : (x, y) \in L,\ y \leqslant a(x)\}$$

and observe that

$$f(x) \leqslant u(x) \leqslant C_1 |x|^{3/2} \qquad \text{for} \quad |s| < \delta \tag{3}$$

because $E \supset L$. Moreover, since $x_0 \in \partial L$, it follows that there is a ball of radius R centered at $(0, -R)$ such that $(x, f(x))$ lies above the ball. That is,

$$f(x) \geqslant \sqrt{R^2 - |x|^2} - R = \frac{-|x|^2}{\sqrt{R^2 - |x|^2} + R} \geqslant -\frac{|x|^2}{R}. \tag{4}$$

Now we claim that there exists $\delta_1 = \delta_1(\delta, C_1, R) < \delta$ such that for $|x_1| \leqslant \delta_1$, the center (a_1, b_1) of the ball whose boundary contains $(x_1, f(x_1))$ satisfies

$$|x_1 - a_1| < \lambda R, \tag{5}$$

where $\lambda = \lambda(\delta_1, C_1, R) < 1$. If this claim were not true, then a sequence $\{x_j\}$ could be found such that $|x_j - a_j| = R$ with $|x_j| \to 0$, where (a_j, b_j) is the center of the ball whose boundary contains $(x_j, f(x_j))$. Since

$$|y - f(x_j)|^2 + |x - a_j|^2 = R^2 \tag{6}$$

by passing to a subsequence if necessary, we find that $\{a_j\}$ must converge to a point $a_0 \in R^{n-1}$ such that

$$|a_0| = R. \tag{7}$$

Hence, by (6) we find that there exists a ball centered at $(a_0, 0)$ whose boundary contains $(0, 0)$. However, the boundary of the ball centered at $(0, -R)$ also contains $(0, 0)$ which contradicts (3), thus establishing the claim.

Now consider any point $x_1 \in R^{n-1}$ such that $|x_1| < \delta_1/4$. Suppose $(x_1, f(x_1))$ lies on the boundary of the ball centered at (a_1, b_1). Note that for x satisfying $|x - x_1| < ((1-\lambda)/2)R$, we have $|x - a_1| \leqslant |x - x_1| + |x_1 - a_1| \leqslant ((1-\lambda)/2)R + \lambda R = \mu R$, where $\mu = (1+\lambda)/2 < 1$. Using the fact that the equation of the upper hemisphere is given by $y = b_1 + \sqrt{R^2 - |x - a_1|^2}$, for x such that $|x - x_1| < ((1-\lambda)/2)R$, we obtain

$$|\nabla y(x)| = \frac{|x - a_1|}{\sqrt{R^2 - |x - a_1|^2}} \leqslant \frac{\mu}{1 - \mu^2} \cdot \frac{1}{R} \equiv C_2, \tag{8}$$

where $C_2 = C_2(C_1, R)$. Also, we have

$$\frac{\partial^2 y}{\partial x_i \partial x_j} = -\frac{\delta_{ij}}{\sqrt{R^2 - |x - a|^2}} - \frac{(x_i - a_i)(x_j - a_j)}{(R^2 - |x - a|^2)^{3/2}}. \tag{9}$$

Thus, from (9) we obtain

$$\left|\frac{\partial^2 y}{\partial x_i \partial x_j}\right| \leqslant \frac{2}{\sqrt{1 - \mu^2}} \cdot \frac{1}{R} + \frac{\mu^2}{(1 - \mu^2)^{3/2}} \cdot \frac{1}{R} \equiv C_3 \tag{10}$$

for $|x-x_1|<((1-\lambda)/2)R$, where $C_3=C_3(C_1, R)$. Hence, for $h\in R^{n-1}$ such that $|h|<\min(\delta_1/4, ((1-\lambda)/2)R))$ we have from Taylor's Theorem, (8), and (10)

$$y(x_1+h)\geqslant y(x_1)-C_2|h|-C_3|h|^2. \tag{11}$$

Then, since $f(x_1+h)\geqslant y(x_1+h)$ while $f(x_1)=y(x_1)$ it follows that

$$f(x_1+h)\geqslant f(x_1)-C_2|h|-C_3|h|^2. \tag{12}$$

This argument can now be repeated starting at x_1+h instead of x_1, where x_1 is is before and $|h|<\min(\delta_1/4, ((1-\lambda)/2)R)$. Note that bounds (8) and (10) are unchanged; thus we obtain

$$f(x_1)\geqslant f(x_1+h)-C_2|h|-C_3|h|^2$$

so that

$$-C_2-C_3|h|\leqslant\frac{f(x_1+h)-f(x_1)}{|h|}\leqslant C_2+C_3|h|$$

thus proving that f is Lipschitz. ∎

LEMMA 2. *There are constants C_1, C_2 depending only on R and the $C^{1,1/2}$-norm of u such that*

$$\frac{1}{|h|^2}[f(x_0+h)+f(x_0-h)-2f(x_0)]\geqslant -C_1$$

$$\frac{1}{|h|^2}[F(x_0+h)+F(x_0-h)-2F(x_0)]\leqslant C_2$$

for all small $|x_0|$ and $|h|$.

Proof. We proceed to establish only the first inequality, the proof of the second being similar. In light of the uniform interior ball condition satisfied by L, we conclude that for each $|x_0|$ sufficiently small, there is a ball $B(R)$ contained within L whose boundary contains $(x_0, f(x_0))$. Observe that this ball contains a ball $B(q, r)$ with $r\leqslant R$ such that $(x_0, f(x_0))\in\partial B(q, r)$ and the center $q=(a, b)$ satisfies $|a|<\delta$. Note also that r depends only on R. Near such a point $(x_0, f(x_0))$, the boundary of the ball can be written as

$$y=-b+\sqrt{r^2-|x-a|^2},$$

and therefore

$$\frac{\partial^2 y}{\partial x_i\,\partial x_j} = -\frac{\delta_{ij}}{\sqrt{r^2-|x-a|^2}} - \frac{(x_i-a_i)(x_j-a_j)}{(r^2-|x-a|^2)^{3/2}},$$

which implies

$$\left|\frac{\partial^2 y}{\partial x_i\,\partial x_j}\right| \leqslant \frac{2}{r} + \frac{8\delta^2}{r}$$

whenever $|x-a| \leqslant \delta r$. Hence, by Taylor's Theorem, we have

$$y(x_0+h) \geqslant y(x_0) + \frac{h\cdot(a-x_0)}{\sqrt{r^2-|x_0-a|^2}} - C_1|h|^2,$$

whenever $|x_0|$, $|x_0+h| \leqslant \delta r$, where C_1 depends on r and the $C^{1,1/2}$-norm of u. Since $f(x_0)=y(x_0)$ and for other values f dominates y, it follows that

$$\frac{1}{|h|^2}\left[f(x_0+h)+f(x_0-h)-2f(x_0)\right] \geqslant -C_1. \quad \blacksquare$$

THEOREM 3. *If E is a minimizer of* (2), *then there is a set* sing ∂E *of Hausdorff dimension at most $n-8$ such that*

(i) *∂E is real analytic in a neighborhood of each point in $\Omega \cap \partial E$–(sing $\partial E \cup L \cup M$).*

(ii) *∂E is $C^{1,1}$-regular in a neighborhood of each point of $\Omega \cap \partial E \cap (\partial L \cup \partial M)$.*

Proof. Since ∂E is area minimizing in $\Omega-(L\cup M)$, part (i) is a consequence of well-known results in minimal surface theory [G].

For the proof of (ii), let $p \in (\Omega \cap \partial E) \cap (\partial L \cup \partial M)$. Then as in Lemma 2, there is a neighborhood of p such that E can be represented as the graph of a $C^{1,1/2}$-function u that lies between the graphs of functions f and F that represent respectively ∂L and ∂M. The function u therefore satisfies a double-obstacle problem for the minimal surface operator. That is, after the introduction of a suitable set of coordinates, so that $u(0)=0$, $Du(0)=0$, there exists $\delta>0$ such that with $B(0,\delta)=U$

$$\int_U \frac{Du\cdot D(v-u)}{\sqrt{1+|Du|^2}}\,dx \geqslant 0 \tag{13}$$

for all $v \in K = C^{0,1}(\bar{U}) \cap \{v: v = u \text{ on } \partial U, f \leqslant v \leqslant F \text{ on } \bar{U}\}$. Furthermore, by Lemma 2 we may assume f and F belong to $C^{0,1}(\bar{U})$ with

$$|Df|, |DF| \leqslant C_3$$

$$\frac{1}{|h|^2}[f(x_0+h)+f(x_0-h)-2f(x_0)] \geqslant -C_1 \tag{14}$$

$$\frac{1}{|h|^2}[F(x_0+h)+F(x_0-h)-2F(x_0)] \leqslant +C_2$$

for all $|x_0|$ and $|h|$ sufficiently small. We now consider a smooth approximation to problem (13) by changing the obstacles slightly. That is, we add a small amount to F, substract a similar amount from f, and then mollify these functions so that the resulting functions f_ε and F_ε satisfy $f_\varepsilon < F_\varepsilon$ in $\bar{U}$. If u_ε denotes locally the graph of the solution to the double-obstacle problem with these mollified obstacles, then $f_\varepsilon \leqslant u_\varepsilon \leqslant F_\varepsilon$ on ∂U. The mollified functions f_ε and F_ε are elements of $C^\infty(U)$ and satisfy (14). If we consider the corresponding double-obstacle problem with f_ε and F_ε replacing f and F, then we know that the corresponding solution u_ε is $C^{1,1}(U)$; cf. [*KS*].

Assuming the validity of Lemmas 4 and 5 below, we obtain *a priori* estimates on the first and second derivatives of u_ε. That is, we have that

$$|Du_\varepsilon| \leqslant C_3$$

and that if $U' \subset\subset U$, then

$$|D_{ij}u_\varepsilon| \leqslant C \qquad \text{a.e. on } U', \tag{15}$$

where C depends only on the $C^{1,1/2}$-norm of u, the constants of (14), and U' but not on any of the other properties of f_ε and F_ε. This is sufficient to establish our result because ∇u is Lipschitz and thus (15) implies that the difference quotients of the first derivatives are uniformly bounded independent of ε. Therefore, by the Arzela–Ascoli compactness theorem, we have (for a subsequence) that u_ε converges uniformly to some function w and similarly for its first derivatives. Clearly, w is a solution of Problem (13). Since any C^1 solution of Problem (13) is unique [KS], it follows that $u = w$. ∎

For the purposes of Lemmas 4 and 5 below, it is sufficient to consider Problem (13) with F and f replaced by F_ε and f_ε. For simplicity of notation, we drop the subscript ε and assume instead that F and f are smooth with $f < F$ in $\bar{U}$. The associated $C^{1,1}$-solution is denoted by u. This brings us to the following Lemma.

LEMMA 4. *Let u be the solution of Problem* (13), *where f and F are assumed to be smooth. Let*

$$A_1 = U \cap \{x: u(x) = f(x)\}$$
$$A_2 = U \cap \{x: u(x) = F(x)\}$$
$$A_3 = U \cap \{x: f(x) < u(x) < F(x)\}.$$

Then, there exists a constant C depending only on the constants in (14) *and the $C^{1,1/2}$-norm of u such that for each second order partial derivative*

(i) $|D_{ij}f| \leqslant C$ *on* A_1,

(ii) $|D_{ij}F| \leqslant C$ *on* A_2,

(iii)

$$\limsup_{\substack{y \to x \\ y \in A_3}} |D_{ij}u(y)| \leqslant C$$

uniformly for $x \in \partial A_3 \cap (A_1 \cup A_2)$.

Proof. First, we prove that if $x_1 \in A_1$, then

$$Mf(x_1) \leqslant 0,$$

where M denotes the mean curvature operator

$$Mv = \operatorname{div}\left(\frac{Dv}{\sqrt{1+|Dv|^2}}\right) = a^{ij}(Dv)\frac{\partial^2 v}{\partial x_i \partial x_j},$$

where

$$a^{ij}(p) = \frac{\delta^{ij}}{\sqrt{1+|p|^2}} - \frac{p_i p_j}{(1+|p|^2)^{3/2}}. \tag{16}$$

Observe that the nonnegative function $u-f$ attains its minimum on A_1 and therefore $\nabla u = \nabla f$ on A_1. Now, if we had $Mf(x_1) > 0$, then we could choose γ, $\eta > 0$ such that

$$a^{ij}(Du)\frac{\partial^2 f}{\partial x_i \partial x_j} \geqslant \gamma > 0 \qquad \text{in } B(x_1, \eta).$$

Here we have used the fact that $Du(x_1) = Df(x_1)$. Let

$$g(x) = f(x) + \varepsilon(\eta^2 - |x - x_1|^2)$$

so that $g=f$ on $\partial B(x_1, \eta)$ and for ε sufficiently small

$$a^{ij}(Du)\frac{\partial^2 g}{\partial x_i \partial x_j} \geqslant 0 \qquad \text{in } B(x_1, \eta).$$

But near x_1, u is a supersolution of the minimal surface equation since we may assume that $u<F$ near x_1. Hence,

$$a^{ij}(Du)\frac{\partial^2 u}{\partial x_i \partial x_j} \leqslant 0 \qquad \text{in } B(x_1, \eta).$$

Since $g=f\leqslant u$ on $\partial B(x_1, \eta)$, we must have $g \leqslant u$ in $B(x_1, \eta)$; cf. [GT, Theorem 9.1], which is a contradiction at x_1. Thus, we have shown that $Mf(x_1)\leqslant 0$.

We now proceed to obtain bounds on the second derivatives of f at a point x_1 of Λ_1. For this purpose, note that (14) implies that for any direction ξ

$$D_{\xi\xi}(f)\geqslant -C_1 \tag{17}$$

and similarly

$$D_{\xi\xi}(F)\leqslant C_2. \tag{18}$$

For any direction ξ, we may assume that ξ corresponds to the x_1-axis and that the other directions are chosen so that

$$D_{ij}f(x_1)=0 \qquad \text{for} \quad i\neq j.$$

Then we have

$$a_{11}(Du)\frac{\partial^2 f}{\partial \xi^2}(x_1)+\sum_{i=2}^{n} a_{ii}(Du)\frac{\partial^2 f}{\partial x_i^2}(x_1)\leqslant 0$$

so that by (14) and (17)

$$\begin{aligned}-C_1 \leqslant \frac{\partial^2 f}{\partial \xi^2} \leqslant -\frac{1}{a_{11}(Du)}\sum_{i=2}^{n} a_{ii}(Du)\frac{\partial^2 f}{\partial x_i^2}(x_1) &\leqslant C_1(n-1)(1+|Du|^2)\\ &\leqslant C_1(n-1)(1+C_3^2).\end{aligned}$$

That is,

$$|D_{\xi\xi}f(x_1)|\leqslant C.$$

Since this is true for any direction ξ we obtain that on Λ_1

$$|D_{ij}f| \leqslant C \tag{19}$$

and similarly on Λ_2

$$|D_{ij}F| \leqslant C. \tag{20}$$

This establishes (i) and (ii).

For the analysis of (iii), we consider the open set Λ_3 and analyze the behavior of the pure second derivatives of u at the boundary. The analysis will be focused on the three sets represented by the decomposition

$$\partial\Lambda_3 = (\partial\Lambda_3 \cap \partial U) \cup (\partial\Lambda_3 \cap \Lambda_1) \cup (\partial\Lambda_3 \cap \Lambda_2).$$

If $x \in \partial\Lambda_3 \cap \Lambda_1$, the main result of [C, Theorem 1] implies that for any direction ξ,

$$\liminf_{\substack{y \to x \\ y \in \Lambda_3}} D_{\xi\xi}(u-f)(y) \geqslant 0$$

uniformly in x, thus establishing

$$\liminf_{\substack{y \to x \\ y \in \Lambda_3}} u_{\xi\xi} \geqslant -C.$$

with C from (19). Since $Mu=0$ in Λ_3 and $|Du| \leqslant C$, it follows from the above inequality (as in the proof of (19)) that

$$\limsup_{\substack{y \to x \\ y \in \Lambda_3}} u_{\xi\xi}(y) \leqslant C.$$

Since this is true for any direction ξ, it follows that

$$\limsup_{\substack{y \to x \\ y \in \Lambda_3}} |D_{ij}u(y)| \leqslant C$$

for $x \in \partial\Lambda_3 \cap \Lambda_1$. A similar result holds for $x \in \partial\Lambda_3 \cap \Lambda_2$ so we now conclude that

$$\limsup_{\substack{y \to x \\ y \in \Lambda_3}} |D_{ij}u(y)| \leqslant C \tag{21}$$

uniformly for $x \in \partial\Lambda_3 \cap (\Lambda_1 \cup \Lambda_2)$. ∎

LEMMA 5. *Let u be a solution of Problem* (13) *where f and F are assumed to be smooth. Then, for each $U' \subset\subset U$ there is a constant C depending only on the constants in* (14) *and the $C^{1,1,2}$-norm of u such that*

$$|D_{ij}u| \leqslant C \qquad \text{almost everywhere on } U'.$$

Proof. Let L denote the operator

$$L\omega = \frac{\partial}{\partial x_i}\,[a^{ij}(Du)\,\omega_{x_j}],$$

where the a^{ij} are defined by (16). In view of the fact that $u \in C^{1,1}(U)$, in particular that $u \in C^1(U)$, this operator is well defined. Moreover, since $|\nabla u| \leqslant C_3$ on U, it follows that

$$\lambda\,|\eta|^2 = a^{ij}(Du)\,\eta_i\eta_j \qquad \text{for} \quad \eta \in R^n$$

and

$$|a^{ij}(Du)| \leqslant M$$

on U, where λ and M are constants depending only on C_3. Now suppose $U' \subset\subset U$ and choose $\eta \in C_0^\infty(U)$ such that $\eta = 1$ on U', $\eta > 0$. We now consider the operator L defined on the open set A_3. Then, with $\omega = u_{\xi\xi}$, which is smooth in A_3, it can be shown (cf. [KS, p. 127])

$$L(\eta u_{\xi\xi}) = \frac{\partial g_i}{\partial x_i} + g_0$$

with

$$g_i = -\eta(a^{ij}(Du))_\xi\, u_{\xi j} + 2a^{ij}\eta_j u_{\xi\xi}$$

$$g_0 = u_{\xi\xi}\,\frac{\partial}{\partial x_i}\,[a^{ij}\eta_j] - \eta_i(a^{ij}(Du))_\xi u_{\xi j}.$$

We intend to invoke [GT, Theorem 8.16] as applied to $L(\eta u_{\xi\xi})$ on proper subdomains of A_3. For this it is necessary to obtain L^p bounds for g_i and g_0. Since $u \in C^{1,1}(U)$, we see that D^2u exists almost everywhere on A_1 in particular. Choose such a point x_1 of differentiability. Since $f < F$ it follows that near x_1, u is a supersolution of the minimal surface equation. Moreover, since x_1 is a point at which $u - f$ assumes a minimum, we have $Du(x_1) = Df(x_1)$. Using both the ellipticity of the matrix $a^{ij}(Du)$ and the fact that the matrix $D^2(u-f)(x_1)$ is nonnegative, we have

$$0 \geqslant a^{ij}(Du)\,D_{ij}u(x_1) \geqslant a^{ij}(Du)\,D_{ij}f(x_1) = a^{ij}(Df)\,D_{ij}f(x_1).$$

By appealing to (19) it follows that $|Mu| \leqslant C$ almost everywhere on A_1. A similar argument establishes the same conclusion on A_2. Since $Mu = 0$ on A_3 we have $|Mu| \leqslant C$ a.e. on U, and we now are able to obtain local L^p estimates on the second derivatives of u in U. Indeed, by [GT, Theorem 9.11], we have that if $1 < p < \infty$ and $V \subset\subset U$ then

$$\|D^2 u\|_{p;V} \leqslant C, \tag{22}$$

where C depends on p, n, V, and the $C^{1,1,2}$-norm of u. Now let

$$A_\varepsilon = A_\varepsilon \cap \{x\colon d(x, \partial A_3) > \varepsilon\}.$$

and

$$U'' = \{x\colon \eta(x) > 0\} \subset\subset U.$$

For sufficiently small $\varepsilon > 0$, it follows from (21) that

$$|D^2 u(y)| \leqslant C + 1$$

for almost every $y \in A_\varepsilon$ uniformly close to $\partial A_\varepsilon \cap (A_1 \cup A_2)$. Since u is analytic in A_3, it follows that $u_{\xi\xi} \in C^\infty(A_\varepsilon) \cap W^{1,2}(A_\varepsilon)$. Hence,

$$\begin{aligned}\sup_{\partial A_\varepsilon} (\eta u_{\xi\xi}) &\leqslant \sup_{\partial U''} (\eta u_{\xi\xi}) + \sup_{\partial A_\varepsilon \cap U''} (\eta u_{\xi\xi}) \\ &\leqslant 0 + (C+1).\end{aligned}$$

In view of (22) we may now appeal to [GT, Theorem 8.16] to conclude that $|\eta u_{\xi\xi}| \leqslant C'$ on A_ε, where C' depends on C from estimate (22), and A_ε. Since this is valid for every directi on ξ, we have that $|D^2 u| \in L^\infty_{\text{loc}}(V \cap A_3)$ whenever $V \subset\subset U$ since $\bigcup_{\varepsilon > 0} A_\varepsilon = A_3$.

We now show that $|D^2 u| \in L^\infty(A_1 \cup A_2)$, which along with $|D^2 u| \in L^\infty_{\text{loc}}(V \cap A_3)$ for every $V \subset\subset A_3$ is sufficient to establish (15) and thus conclude the proof. Note that $\nabla u = \nabla f$ everywhere on A_1. Since ∇u is Lipschitz, it is differentiable almost everywhere. Moreover, $\nabla u = \nabla f$ everywhere on A_1 and since almost every point of A_1 is a point of linear metric density [S, p. 298], we may conclude that $D^2 u = D^2 f$ almost everywhere on A_1. Hence, $|D_{ij} u| \in L^\infty(A_1)$ from (19). Similarly, we conclude that $|D_{ij} u| \in L^\infty(A_2)$. ∎

References

[C] L. Caffarelli, The regularity of free boundaries in higher dimensions, *Acta Math.* **139** (1978), 155–184.

[CR] L. Caffarelli and N. Riviere, On the smoothness and analyticity of free boundaries in variational inequalities, *Ann. Scuola Norm. Sup. Pisa* **3** (1976), 289–320.

[GT] D. GILBARG AND N. TRUDINGER, "Elliptic Partial Differential Equations of Second Order," 2nd ed., Springer, New York, 1983.

[G] E. GIUSTI, "Minimal Surfaces and Functions of Bounded Variation," Birkhäuser, Basil, 1985.

[KS] D. KINDERLEHRER AND G. STAMPACCHIA, "An Introduction to Variational Inequalities and Their Applications," Academic Press, San Diego, 1980.

[KoS] R. V. KOHN AND G. STRANG, The constrained least gradient problem, *in* "Non-classical Continuum Mechanics" (R. Knops and A. Lacy, Eds.), pp. 226–243, Cambridge Univ. Press, London/New York, 1987.

[S] S. SAKS, "Theory of the Integral," Warsaw, 1937.

[SWZ] P. STERNBERG, G. Williams, and W. Ziemer, The constrained least gradient problem in R^n, to appear.

[T] I. TAMANINI, Boundaries of Caccioppoli sets with Holder-continuous normal vector, *J. Reine Angew. Math.* **334** (1982), 27–39.